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LETTERS TO THE EDITOR

APPLICATION OF SECONDARY BIFURCATIONS TO LARGE-AMPLITUDE LIMIT CYCLES IN MECHANICAL SYSTEMS

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1. INTRODUCTION

Periodic solutions of autonomous non-linear dynamical systems are called limit cycles. Limit cycles are also described as self-excited oscillations to distinguish them from oscillations induced by a time-dependent forcing function. Mathematically, limit cycles of an autonomous non-linear dynamical system

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mu), \tag{1}$$

where $\mathbf{x}$ is the state vector, and μ is a scalar parameter, are solutions such that

$$\mathbf{x}(t + T) = \mathbf{x}(t), \tag{2}$$

where T is the period of oscillation. Limit cycles appear as closed orbits in the state space of the elements of the vector $\mathbf{x}$. Limit cycles have been observed in a wide variety of biological, chemical, electrical, and mechanical systems [1].

In dynamical systems of the form of equation (1), limit cycles commonly arise at a Hopf bifurcation of an equilibrium solution with varying values of the parameter μ [2]. The stability of an equilibrium solution is indicated by the eigenvalues of the Jacobian matrix of the function f in equation (1) evaluated at the equilibrium solution. All eigenvalues need to lie in the left half complex plane, i.e., have negative real part, for stability. When a pair of complex conjugate eigenvalues cross over from the left half complex plane to the right, with varying values of parameter μ, then the associated equilibrium solution loses stability. The critical value of μ for which the eigenvalues lie precisely on the imaginary axis corresponds to the onset of instability, and is called a Hopf bifurcation point.

At a Hopf bifurcation point, either stable limit cycles are created about the unstable equilibrium solutions (supercritical Hopf bifurcation), or unstable limit cycles are created about the stable equilibrium solutions (subcritical Hopf bifurcation). These two types of Hopf bifurcation are illustrated in Figure 1, where the limit cycles are represented by the maximum amplitude of oscillation. It can be noticed in Figure 1 that the amplitude of the limit cycles builds up gradually as one moves away from the Hopf bifurcation point. The precise rate of build-up of the amplitude is given by the Hopf bifurcation theorem [2].

There are systems where, with varying values of the parameter μ, limit cycles of finite amplitude are abruptly encountered on crossing the Hopf bifurcation point. In some other systems, limit cycles created according to the Hopf bifurcation theorem suddenly give way to limit cycles with much larger amplitudes. Such limit cycles, whose amplitude does not build up gradually away from the Hopf bifurcation point, are called large-amplitude limit cycles. Since unstable limit cycles are not physically observed, we shall be interested only in stable large-amplitude limit cycles.

Stable large-amplitude limit cycles may be created in one of the following two ways. The family of unstable limit cycles arising at a subcritical Hopf bifurcation could undergo a

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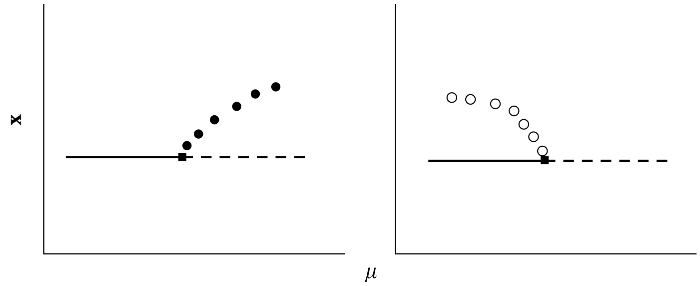

Figure 1. Creation of stable limit cycles at a supercritical Hopf bifurcation, and unstable limit cycles at subcritical Hopf bifurcation: ——, stable equilibria; - - - , unstable equilibria; ●, stable limit cycles; ○, unstable limit cycles; ■, Hopf bifurcation.

fold bifurcation at which the unstable limit cycles turn stable. This is sketched in Figure 2 where, with increasing values of the parameter μ, the system jumps at the Hopf bifurcation point from the stable equilibrium solution to the stable large-amplitude limit cycle family, as indicated by the upward arrow. For decreasing values of μ, the reverse jump from the stable limit cycle family to the stable equilibrium solution occurs at the fold bifurcation, as shown by the downward arrow. Thus, the presence of large-amplitude limit cycles gives rise to hysteresis in the response of the system with varying values of the parameter μ. Periodic oscillations in axial flow compressors, called surge, have been described as large-amplitude limit cycles of this nature [3].

In the other possibility shown in Figure 2, the family of stable limit cycles arising at a supercritical Hopf bifurcation first undergoes a fold bifurcation where they lose stability, and then undergoes a second fold bifurcation where they regain stability. In this case, with increasing values of μ, the system jumps at the first fold bifurcation to the family of stable large-amplitude limit cycles, and with decreasing values of μ, the system jumps off the large-amplitude limit cycle family to the stable equilibrium solutions at the second fold bifurcation. The system response once again shows a hysteresis behaviour with varying values of μ. Periodic oscillations in aircraft flight dynamics, called wing rock, have been described as large-amplitude limit cycles of this nature [4].

The Hopf bifurcations in Figure 2 are called primary bifurcations. Bifurcations of a solution branch that arises at a primary bifurcation are referred to as secondary bifurcations. Thus, the fold bifurcations in Figure 2 are secondary bifurcations. The onset of limit cycles at a primary bifurcation, as in Figure 1, is gradual. If the limit cycles represent an undesirable operating state, this allows the operator, e.g., the pilot in the case of an aircraft, to take preventive action. In contrast, the onset of large-amplitude limit

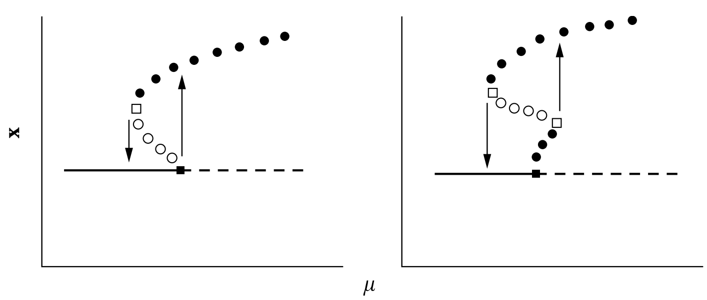

Figure 2. Two possible ways in which large-amplitude limit cycles may be created: —, stable equilibria; - - - , unstable equilibria; ●, stable limit cycles; ○, unstable limit cycles; ■, Hopf bifurcation; □, fold bifurcation.

cycles in Figure 2 is catastrophic, making preventive action by the operator difficult. Also, the hysteresis in the response ensures that the system does not return to the stable equilibrium state at the value of the parameter where the large-amplitude limit cycles originated. Thus, prediction and prevention of large-amplitude limit cycles is a major concern in many practical systems. This requires a characterization of the secondary bifurcations in Figure 2.

The Hopf and fold bifurcations by themselves have been well characterized in terms of conditions for their occurrence, and normal forms [2]. However, in the context of large-amplitude limit cycles, what is required is not a characterization of the Hopf and fold bifurcations separately, but of a primary Hopf–secondary fold bifurcation pair. Of particular interest is the derivation of low-order models that capture the essential dynamics of large-amplitude limit cycles as shown in Figure 2.

In this paper, low-order models for large-amplitude limit cycles are put forward that are minimal in the sense that they contain the least number of non-linear terms to successfully capture the phenomenon of interest. Adopting a constructive approach, we begin with a minimal model for the primary Hopf bifurcation, and progressively incorporate additional terms in the search for a minimal model that also includes a secondary fold bifurcation. The models thus obtained reveal one possible mechanism for the generation of large-amplitude limit cycles.

The models are applied to derive useful results for the two problems described earlier [3, 4] representing the two cases of Figure 2. In view of the applications of interest, only mechanical systems are considered [2], i.e., dynamical systems in which the elements of the state vector occur in pairs, representing a displacement and its associated velocity. Thus, the lowest order models possible are two-dimensional. Mechanical systems arise naturally when motion is described by the differential equations of classical mechanics.

2. THEORY

The van der Pol oscillator has long been considered as a paradigm for limit cycling systems [5]:

$$\ddot{\mathbf{x}} + (\mu_2 \mathbf{x}^2 - \mu_0)\dot{\mathbf{x}} + \mathbf{x} = 0, \tag{3}$$

where $\mu_2 = \pm 1$ for convenience. The only parameter in this model is the linear damping coefficient μ_0. For negative μ_0, the system is damped, and the only equilibrium point at $\mathbf{x} = 0$ is stable. For positive μ_0, the system is undamped, and the equilibrium point at $\mathbf{x} = 0$ is unstable. The onset of instability occurs at $\mu_0 = 0$. It can be shown that, as μ_0 varies from negative values to positive values through zero, a pair of complex conjugate eigenvalues representing the linearized system about the equilibrium point $\mathbf{x} = 0$, cross over from the left half complex plane into the right half. This signals a Hopf bifurcation with the accompanying family of limit cycles. For $\mu_2 = 1$, the Hopf bifurcation is supercritical, and the family of limit cycles is stable. For $\mu_2 = -1$, a family of unstable limit cycles is created at a subcritical Hopf bifurcation.

For this simple model, the limit cycle amplitude can be deduced by observing the coefficients of the terms in $\dot{\mathbf{x}}$. For $\mu_2 = 1$, the effect of the non-linear damping coefficient $\mu_2\mathbf{x}^2$ is to provide positive damping. For sufficiently large $\mathbf{x}$, this counteracts the linear undamping due to $\mu_0 > 0$. Likewise, for $\mu_2 = -1$, the non-linear damping term opposes the linear damping when $\mu_0 < 0$. Hence, the limit cycle amplitude as a function of μ_0 is a direct result of the opposite tendencies of the two terms making up the damping coefficient in equation (3), and is given by $\pm\sqrt{|\mu_0|}$. We are now interested in augmenting the model equation (3) to show large-amplitude limit cycle behaviour. The bifurcation

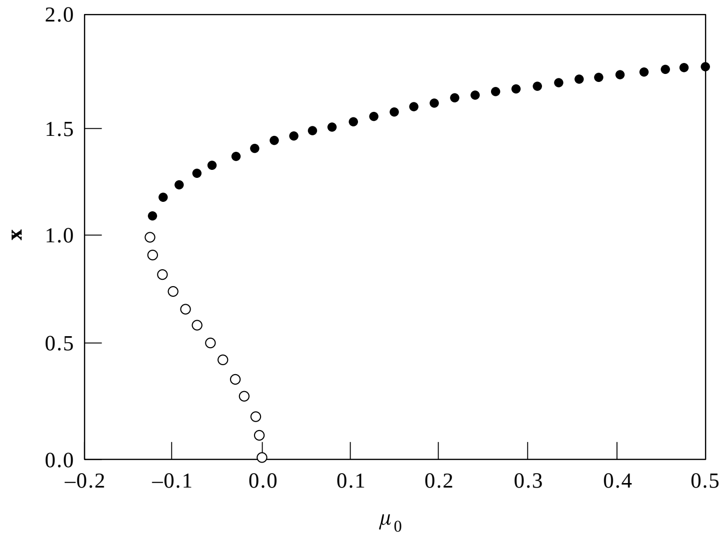

Figure 3. Limit cycle solutions of equation (4): ●, stable limit cycles; ○, unstable limit cycles.

diagrams for large-amplitude limit cycles in Figure 2 show multiple limit cycles over certain ranges of the parameter μ. From the preceding discussion, which connects the limit cycle amplitude with the non-linear damping terms, one way of producing multiple limit cycles could be to augment equation (3) by including damping terms having higher powers of $\mathbf{x}$ as coefficients. Odd-powered polynomial terms are not physically admissable, and are therefore not considered in the augmentation schemes.

We first augment the model equation (3) showing a subcritical Hopf bifurcation for $\mu_2 = -1$ with an additional term $\mu_4\mathbf{x}^4\dot{\mathbf{x}}$. The augmented model appears as

$$\ddot{\mathbf{x}} + (\mu_4\mathbf{x}^4 + \mu_2\mathbf{x}^2 - \mu_0)\dot{\mathbf{x}} + \mathbf{x} = 0. \tag{4}$$

Choosing $\mu_4 = 1$ and $\mu_2 = -1$, a continuation algorithm [6] is used to trace out the family of limit cycles emerging at the subcritical Hopf bifurcation point $\mu_0 = 0$. Results of the computation are plotted in Figure 3, which shows that the family of unstable limit cycles undergoes a fold bifurcation leading to stable large-amplitude limit cycles. The location of the fold bifurcation depends on the value of $\mu_4 > 0$. Thus, equation (4) provides a minimal model for the large-amplitude limit cycles of the first type in Figure 2.

Next, the model equation (3) with $\mu_2 = 1$ is augmented by the fourth order damping term $\mu_4\mathbf{x}^4\dot{\mathbf{x}}$. This augmented model looks identical to equation (4), but shows a supercritical Hopf bifurcation at $\mu_0 = 0$. For $\mu_4 < 0$, the stable limit cycles that emerge at the Hopf bifurcation turn unstable at a fold bifurcation. However, it is seen that the non-linear damping terms in this model are unable to induce a second fold bifurcation, and hence cannot show the large-amplitude limit cycles of the second type in Figure 2. Continuing with the same logic as before, we further augment this model with a non-linear damping term of sixth order $\mu_6\mathbf{x}^6\dot{\mathbf{x}}$. The model now appears as

$$\ddot{\mathbf{x}} + (\mu_6\mathbf{x}^6 + \mu_4\mathbf{x}^4 + \mu_2\mathbf{x}^2 - \mu_0)\dot{\mathbf{x}} + \mathbf{x} = 0. \tag{5}$$

The family of limit cycles originating at the Hopf bifurcation of equation (5) for $\mu_2 = 1$, $\mu_4 = -1$ and $\mu_6 = 0{\cdot}2$ is computed using the continuation algorithm employed previously [6]. The output, plotted in Figure 4, shows a pair of fold bifurcations of the limit cycle family, and the resulting large-amplitude limit cycles. It is seen that the locations of the

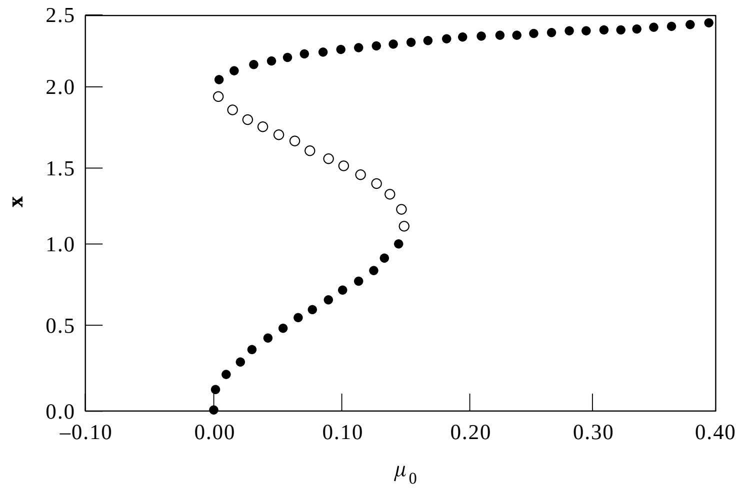

Figure 4. Limit cycle solutions of equation (5). ●, stable limit cycles; ○, unstable limit cycles.

fold bifurcations depend on the values of the parameters μ_4 and μ_6. Thus, equation (5) provides a minimal model for the large-amplitude limit cycles of the second type in Figure 2.

3. APPLICATIONS

3.1. *Surge oscillations in axial flow compressors*

Surge is a large-amplitude periodic oscillation of the mass flow along the axis of the compressor. It has been shown by Abed *et al*. [3] that the onset of surge occurs at a subcritical Hopf bifurcation in the manner of Figure 3. Axial compressors in modern aircraft gas turbine engines frequently encounter surge since they tend to operate close to the point of onset of surge in order to maximize the pressure rise across the compressor for a given rotational speed. Therefore, there is a need to a predict the occurrence of surge in axial flow compressors.

A reduced two-dimensional model for the non-dimensional mass flow through the compressor $\dot{m}_c$ was put forward by Greitzer [7] as follows

$$\frac{\mathrm{d}^2\dot{m}_c}{\mathrm{d}t^2} + \left[\frac{1}{BF'(\dot{M})} - BC'(\dot{m}_c)\right]\frac{\mathrm{d}\dot{m}_c}{\mathrm{d}t} + \left[\dot{m}_c - \dot{M} + \frac{F(\dot{M}) - C(\dot{m}_c)}{F'(\dot{M})}\right] = 0, \tag{6}$$

where B is a parameter, $\dot{M}$ is a reference non-dimensional mean flow, $F(\dot{M})$ and $F'(\dot{M})$ are constants, and $C(\dot{m}_c)$ is the compressor characteristic. The compressor characteristic relates the pressure rise across the compressor to the mass flow through it. This is usually a highly non-linear relationship that must be derived from experiments. It is found that surge limit cycles are strongly dependent on the form of the compressor characteristic. This raises the question of what is the correct function to fit to the experimentally determined compressor characteristic for the prediction of surge.

It may be observed from equation (6) that the damping coefficient depends on $C'(\dot{m}_c)$, the derivative of the compressor characteristic. From the theory developed in the previous section, the minimal model for the large-amplitude limit cycles of Figure 3 contains non-linear damping terms of the fourth order. This implies that $C'(\dot{m}_c)$ must contain terms of the fourth order in $\dot{m}_c$, and therefore, the compressor characteristic needs to be a fifth-order polynomial function. A cubic compressor characteristic was tested by Abed *et al*. [3] and was not found to predict surge. Using the theory of secondary bifurcations, it is now possible to point out that at least a fifth order polynomial fit is required for the experimental data in this case to predict surge.

3.2. *Wing rock oscillations in aircraft flight dynamics*

Wing rock is a limit cycle oscillation in roll and yaw that has been observed on many aircraft at moderate-to-high angles of attack [8]. The onset of wing rock occurs at a supercritical Hopf bifurcation of the four-dimensional lateral equations [4] with increasing values of angle of attack as the parameter. Large-amplitude wing rock has been described by Ananthkrishnan and Sudhakar [4] where the family of limit cycles at the supercritical Hopf bifurcation undergoes a pair of fold bifurcations, as in Figure 4. A six-dimensional coupled lateral–longitudinal model has been suggested for this purpose, and computations with this six-dimensional model have indeed revealed large-amplitude wing rock [4]. Prevention of large-amplitude wing rock is a matter of significant concern.

One problem of interest is to figure out the physical mechanism responsible for large-amplitude wing rock in the model of reference [4]. Strategies for prevention of large-amplitude wing rock are expected to depend on the mechanism responsible for causing it. The relevant two-dimensional equation for the side-slip angle β can be extracted from the six-dimensional model [4] as follows:

$$\ddot{\beta} = (c_0 + c_2\beta^2)\dot{\beta} + (\text{stiffness terms}) + (\text{terms involving other variables}), \tag{7}$$

where c_0 and c_2 are functions of the angle of attack parameter. According to the minimal model developed in the previous section, large-amplitude limit cycles, as in Figure 4, require non-linear damping terms of fourth and sixth orders. However, the non-linear damping term of highest order in equation (7) is quadratic. Hence, it can be concluded that the mechanism causing large-amplitude wing rock in aircraft is not non-linear damping.

4. CONCLUSION

Large-amplitude limit cycles have been described in terms of primary Hopf–secondary fold bifurcation pairs. Minimal models for two cases of large-amplitude limit cycles have been developed for mechanical systems. The mechanism responsible for large-amplitude limit cycles in these models is non-linear damping. The theory is applied to the problems of surge oscillations in axial flow compressors, and wing rock oscillations in aircraft flight dynamics, to derive new and useful results.

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